Bound states of spatial optical dark/gray solitons in nonlocal media

Shigen Ouyang, Wei Hu, and Qi Guo*
Laboratory of Photonic Information Technology, South China Normal University Guangzhou, 510631, P. R. China

(Dated: January 20, 2013)

It is shown that three or more dark/gray solitons can form bound states in nonlocal media. More over dark/gray solitons can form bound states in several balance distances. Numerical simulations indicate that some of such bound states are unstable and will decay into a group of fundamental solitons, while others may be stable. There exist degenerate bound states with the same velocity, Hamiltonian, particle numbers and momentum but decaying in different ways and having different lifetimes.

PACS numbers: 42.65.Tg , 42.65.Jx , 42.70.Nq , 42.70.Df

I. INTRODUCTION

As were pointed out previously by N. I. Nicolov, et. al.[1] and Y. V. Kartashov, et. al.[2], two dark/gray solitons can form bound states due to their long-range attraction in nonlocal nonlinear media. In this paper we will show that three or more dark/gray solitons can form bound states also. As were indicated previously[3], nonlocal dark/gray solitons have exponentially decaying oscillatory tails which in turn give rise to widely distributed exponentially decaying oscillatory light-induced perturbed refractive index. As a result, in this paper, we will find that dark/gray solitons can form bound states in several balance distances. Numerical simulation shows that fundamental dark/gray solitons are stable. Some of such bound states are unstable and will decay into a group of fundamental solitons, while others may be stable. Bound states of no central symmetry or asymmetry will have degenerate states with the same non-vanishing velocity, Hamiltonian, particle numbers and momentum. Some of such degenerate states are unstable and will decay in different ways and have different lifetimes.

II. BOUND STATES OF MULTIPLE DARK/GRAY SOLITONS

The propagation of a paraxial optical beam in a medium with a self-defocusing spatially nonlocal nonlinearity can be described by the dimensionless nonlocal nonlinear Schrödinger equation(NNLSE)[1–3]

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}-u\int R(|x-x'|)|u(x',z)|^2\mathrm{d}x'=0 \quad (1)$$

where $u(x,z)$ is the complex amplitude envelop of the light beam, $I(x,z)=|u(x,z)|^2$ is the light intensity, x and z are transverse and longitude coordinates respectively, $R(x)$ is the real nonlocal response function and satisfies the normalization condition $\int R(x)\mathrm{d}x=1$, $n(x,z)=-\int R(|x-x'|)|u(x',z)|^2\mathrm{d}x'$ is the light-induced perturbed refractive index. Note that not stated otherwise all integrals in this paper will extend over the whole x axis. It is easy to prove if $u(x,z)$ satisfies Eq. (1) with a nonlocal response function $R(x)$, the scale-transformed function $\widetilde{u}(x,z)=\frac{1}{\alpha}u(\frac{x}{\alpha},\frac{z}{\alpha^2})$ satisfies Eq. (1) with another nonlocal response function $\widetilde{R}(x)=\frac{1}{\alpha}R(\frac{x}{\alpha})$, where α is an arbitrary positive real number. So it is adequate to consider dark/gray soliton having asymptotic behavior $|u(x,z)|^2\xrightarrow{x\to\pm\infty}1$. Not stated otherwise all dark/gray solitons in this paper have this asymptotic behavior.

By introducing a transformation $u(x,z)=\psi(x,z)\mathrm{e}^{-iz}$, equation (1) turns into

$$i\frac{\partial\psi}{\partial z}+\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}-\psi\left[\int R(|x-x'|)|\psi(x')|^2\mathrm{d}x'-1\right]=0 \quad (2)$$

which has three integrals of motion[4, 5], namely, the number of particles

$$N=\int(1-|\psi|^2)\mathrm{d}x, \quad (3)$$

the momentum

$$P=\frac{i}{2}\int\left(\psi\frac{\partial\psi^*}{\partial x}-\psi^*\frac{\partial\psi}{\partial x}\right)\left(1-\frac{1}{|\psi|^2}\right)\mathrm{d}x, \quad (4)$$

and the Hamiltonian

$$H=\frac{1}{2}\int\left|\frac{\partial\psi}{\partial x}\right|^2\mathrm{d}x+\int(1-|\psi|^2)\mathrm{d}x+\frac{1}{2}\iint R(|x-x'|)\left[|\psi(x)|^2|\psi(x')|^2-1\right]\mathrm{d}x\mathrm{d}x' \quad (5)$$

In this paper we numerically solve Eq. (2) to find dark/gray soliton solutions

$$\psi(x,z)=\phi(x-vz)\mathrm{e}^{i\theta(x-vz)}, \quad (6)$$

where v is the velocity of the dark/gray soliton relative to the cw background, and real functions ϕ and θ asymptotically approach $\phi(x)\xrightarrow{x\to\pm\infty}1$, $\theta(x)\xrightarrow{x\to\pm\infty}\mp\theta_0$, where θ_0 is a real constant. By inserting Eq. (6) into (2), we get

$$\theta'=v\left(1-\tfrac{1}{\phi^2}\right) \quad (7a)$$

$$\tfrac{\phi''}{2}+\tfrac{v^2+2}{2}\phi-\tfrac{v^2}{2\phi^3}-\phi\int R(x-x')\phi^2(x')dx'=0 \quad (7b)$$

*Electronic address: guoq@scnu.edu.cn

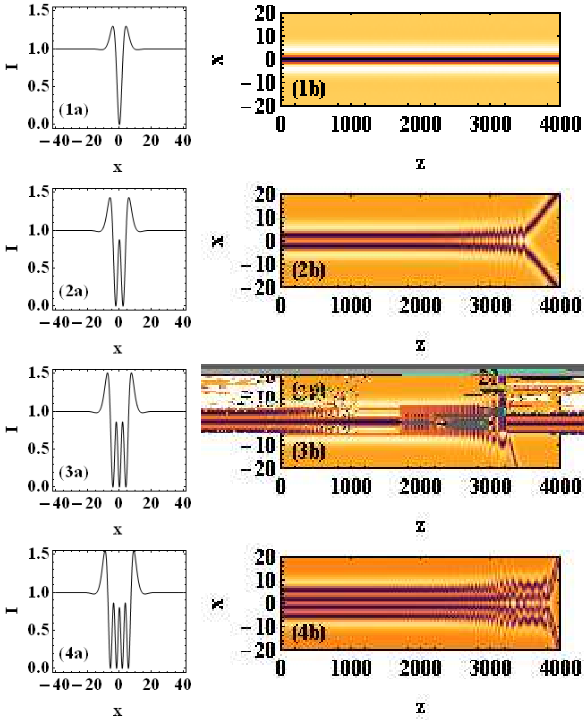

FIG. 1: (1a),(2a),(3a),(4a) are the intensity profiles of ψ_0,ψ_1,$\psi_{1,1}$,$\psi_{1,1,1}$ respectively. (1b),(2b),(3b),(4b) are their counterpart evolutions. Here $w = 5, v = 0$.

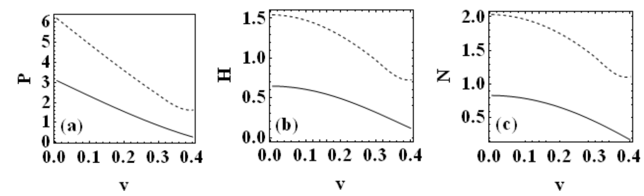

FIG. 2: Functional dependence of momentum P, Hamiltonian H and the number of particles N on the velocity v of ψ_0 (solid line) and ψ_1 (dashing line) when $w = 5$.

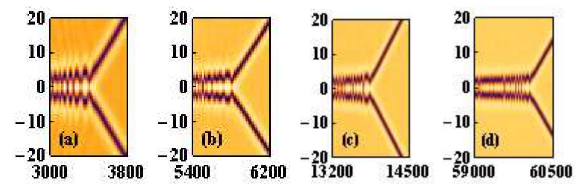

FIG. 3: It needs a longer and longer propagation distance for the decaying of bound states ψ_1 as the characteristic nonlocal response length decreases from (a) $w = 4$ to (b) $w = 2$, to (c) $w = 1.5$, and to (d) $w = 1.2$.

As an example, we consider this following nonlocal case in which the light-induced perturbed refractive index is governed by

$$n - w^2 \frac{\partial^2 n}{\partial x^2} = -|u|^2, \tag{8}$$

which results in $n(x, z) = -\int R(|x - x'|)|u(x', z)|^2 \mathrm{d}x'$, where $R(|x|) = \frac{1}{2w} \exp\left(-\frac{|x|}{w}\right)$ and w is the characteristic nonlocal response length of the media. Numerically solving Eqs. (7), we can find the dark and gray soliton solutions. The intensity profiles of dark solitons ψ_0 and $\psi_1, \psi_{1,1}, \psi_{1,1,1}$ and their counterpart evolutions are shown in Fig. (1) when $w = 5, v = 0$. (Similarly, bound states $\psi_1, \psi_{1,1}, \psi_{1,1,1}$ can form on the interface of two meida[9]). No observable changes in the intensity profile of the fundamental soliton ψ_0 can be found during its propagation . Numerical simulation (not shown here) indicate that an initially broadened beam $\psi(x, 0) = \psi_0(\frac{x}{1.5}, 0)$ will converge into ψ_0 quickly during its propagation. So the fundamental soliton ψ_0 is stable. However, from Fig. (1), with no initial perturbation bound states $\psi_1, \psi_{1,1}, \psi_{1,1,1}$ are all unstable and will decay into a group of fundamental solitons. Numerical simulations (not shown here) also indicate that initially broadened bound states, instead of converging into bound solitary states, will ultimately decay into a group of fundamental solitons.

As has been shown previously[3],when $w > 1/2$ the maximal velocity of gray soliton is $v_{\max} = \sqrt{\frac{4w-1}{4w^2}}$. So when $w = 5$, we have $v_{\max} = 0.436$. As shown in Fig. (4)(1b), when $w = 5$ the bound state ψ_1 with an initial velocity $v = 0.4$ is unstable. Numerical simulations (not shown here) also indicate that ψ_1 with other velocities $v = 0.1, 0.2, 0.3$ are all unstable. So we strongly guest bound state ψ_1 with any velocity are unstable when $w = 5$. Similarly, bound state $\psi_{1,1}, \psi_{1,1,1}$ with any velocity are all unstable when $w = 5$. When $w = 5$, the functional dependence of momentum P, Hamiltonian H and particle numbers N on the velocity v of ψ_0 and ψ_1 are shown in Fig. (2), from which, both for ψ_0 and ψ_1, we get $\mathrm{d}P/\mathrm{d}v < 0$ for all velocity. ψ_0 is stable but ψ_1 is not. So the criterion of dark soliton instability $\mathrm{d}P/\mathrm{d}v > 0$[6–8] may be a sufficient but not necessary condition. Namely we cannot tell a dark/gray soliton whether stable or not if $\mathrm{d}P/\mathrm{d}v < 0$.

As is shown in Fig. (3), it needs a longer and longer propagation distance for bound states ψ_1 with $v = 0$ to decay into fundamental solitons as the characteristic nonlocal response length w decreases from $w = 4$ to $w = 2$, and to $w = 1.5$, and to $w = 1.2$. So it is possible that bound states ψ_1 could be stable for a small enough value of w. But limited by the nature of the numerical simulations method used by this paper we cannot present an exact proof for the stability of such bound state. So it is still an open question of the stability of bound states of dark/gray solitons for small value of w.

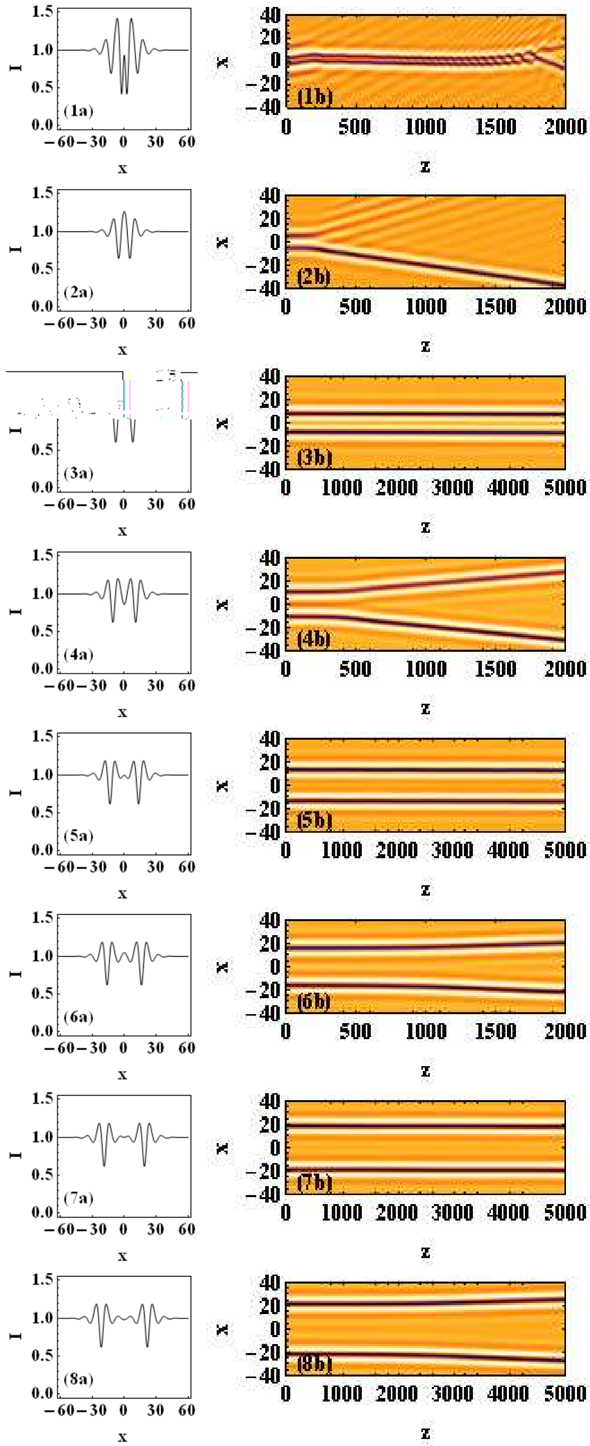

FIG. 4: (1a),(2a),(3a),(4a),(5a),(6a),(7a),(8a) are the intensity profiles of ψ_1,ψ_2,ψ_3,ψ_4,ψ_5,ψ_6,ψ_7,ψ_8 respectively. (2b),(3b),(4b),(5b),(6b),(7b),(8b) are their counterpart evolutions figured in frames moving with velocity $v = 0.4$ and (1b) figured in a frame moving with $v = 0.38$. Here $w = 5$ and velocities of bound states are all $v = 0.4$.

III. BALANCE DISTANCES BETWEEN DARK/GRAY SOLTIONS

As was indicated previously[3], due to the nonlocal nonlinear response of the media the dark/gray solitons have exponentially decaying oscillatory tails when $|v| \le \sqrt{\frac{4w-1}{4w^2}}$ for $w > \frac{1}{4}$. Let $\phi(x) = 1 - \chi(x)$, then when $|x| \to \infty$, we have[3]

$$\chi(x) \approx c_1 \exp(-\lambda_1|x|)\cos(\lambda_2|x| + c_2) \tag{9}$$

where c_1 and c_2 are two constants, and

$$\lambda_1 = \sqrt{\frac{1 - 4w^2v^2 + 4w\sqrt{1-v^2}}{4w^2}}, \tag{10}$$

$$\lambda_2 = \sqrt{\frac{-1 + 4w^2v^2 + 4w\sqrt{1-v^2}}{4w^2}}, \tag{11}$$

and $2\pi/\lambda_2$ is the oscillatory spatial period. As shown in Fig. (5)(a), when $w = 5, v = 0.4$, the fundamental soliton χ_0 has a serial of maximums and minimums interdigitally located at $x_0 = 0, x_1 = 4.57, x_2 = 10.29, x_3 = 15.61, x_4 = 21.10, x_5 = 26.52, x_6 = 31.97, x_7 = 37.41, x_8 = 42.85$. Such exponentially decaying oscillatory tails in turn give rise to an exponentially decaying oscillatory light-induced perturbed refractive index, and dark/gray solitons can form bound states in not only one but several balance distances. For example, as shown in Fig. (4), when $w = 5, v = 0.4$, the balance distances (the distance between two deepest dips) of bound states ψ_1,ψ_2,ψ_3,ψ_4,ψ_5,ψ_6,ψ_7,ψ_8 are d_i = 4.88,10.48,15.97,21.33,26.82,32.24,37.69,43.13, where index i runs from 1 to 8. Obviously we have $d_i \approx x_i$. Let $\Delta d_i \equiv d_{i+1} - d_i$, we have Δd_i = 5.6,5.49,5.36,5.49,5.42,5.45,5.44. On the other hand, from Eq. (11), the half of the oscillatory spatial period $\pi/\lambda_2 = 5.44$ very closes to Δd_i. Other cases with different w and v also show the same relation between π/λ_2 and Δd_i. To study the interaction of two fundamental solitons ψ_0 separated by a distance d, we introduce a coupled state $\psi_d(x) = \phi_d(x)\exp[i\theta_d(x)]$, where

$$\phi_d(x) = 1 - \chi_0\left(x - \tfrac{d}{2}\right) - \chi_0\left(x + \tfrac{d}{2}\right), \tag{12a}$$

$$\theta_d' = v\left(1 - \tfrac{1}{\phi_d^2}\right), \tag{12b}$$

and the index d of ψ_d denotes the distance. Obviously, when $d \to \infty$, there will be no overlap between $\chi_0(x - \frac{d}{2})$ and $\chi_0(x + \frac{d}{2})$, and ψ_d will decouple into two well separated fundamental solitons ψ_0. We note, when $d = x_1, x_3, x_5, \cdots$, the maximums of $\chi_0(x - \frac{d}{2})$ will overlap with the minimums of $\chi_0(x + \frac{d}{2})$, and as Fig. (5)(b) shows, the Hamiltonian of ψ_d takes the minimums, while when $d = x_2, x_4, x_6, \cdots$, the maximums of $\chi_0(x - \frac{d}{2})$ will overlap with the maximums of $\chi_0(x + \frac{d}{2})$, and the Hamiltonian of ψ_d takes the maximums. On the other hand, as indicated by Fig. (5)(c),(d), the difference between ψ_3

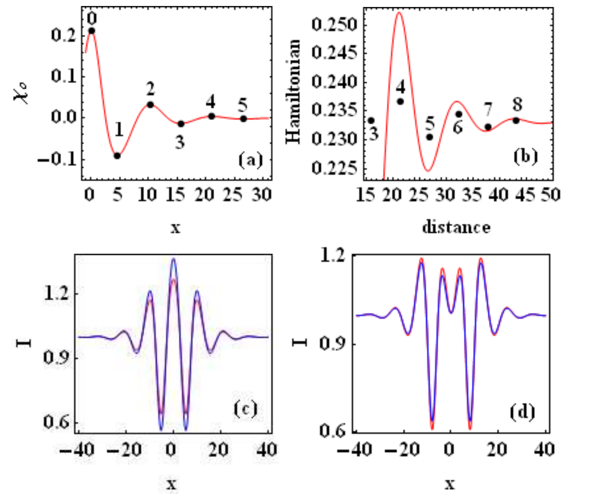

FIG. 5: (a)fundamental soliton $\chi_0(x)$ (red line) and its maximums and minimums (black dots). (b) the Hamiltonian of ψ_d (red line) and those of $\psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8$ (black dots). (c),(d) the intensity profiles of ψ_2, ψ_3 (red line) and ψ_{d_2}, ψ_{d_3} (blue line). Here $w = 5, v = 0.4$.

and ψ_{d_3} is small, and the difference between ψ_i and ψ_{d_i} will decrease as d_i increases. So, as Fig. (5)(b) shows, though there is a difference between the Hamiltonian of ψ_{d_i} and those of ψ_i, the Hamiltonian of ψ_i assumes a similar behavior of that of ψ_d, which may qualitatively explain why $d_i \approx x_i$ and $\Delta d_i \approx \pi/\lambda_2$. Numerical simulations shown in Fig. (4) indicate that $\psi_2, \psi_4, \psi_6, \psi_8$ are all instable, while ψ_3, ψ_5, ψ_7 may be all stable, and ψ_1 is weakly instable when $w = 5, v = 0.4$.

Numerical simulations with initially broadened bound states with vanishing velocity $v = 0$, seeing Fig. (6), also indicate that slightly initial departure from the bound states ψ_2 and ψ_4 can result in a great different evolutions in the long run. ψ_2 and ψ_4 are both instable when $w = 5, v = 0$ also. On the other hand the initially broadened bound state of ψ_3, instead of converging into ψ_3 or decaying into two fundamental solitons, will fall in a seemingly eternal (over a distance larger than $z = 100000$) vibrations around ψ_3. So bound state ψ_3 may be oscillatory-stable. But it needs a more rigorous proof beyond the numerical simulation method of this paper to judge the stability of ψ_3, ψ_5, ψ_7.

Since dark/gray solitons can form bound states at several balance distances, we can construct bound states like $\psi_{1,2,1}$ or $\psi_{\psi_{3,3}}$, and so on. Interestingly, as shown in Fig. (7), bound states with no central symmetry or asymmetry will have two degenerate states, like $\psi_{1,2}$ and $\psi_{2,1}$, moving with the same non-vanishing velocity and having the same momentum, Hamiltonian and the particle numbers but decaying in different ways and having different life-times. However, there may still exist stable degenerate states, like $\psi_{3,5}$ and $\psi_{5,3}$.

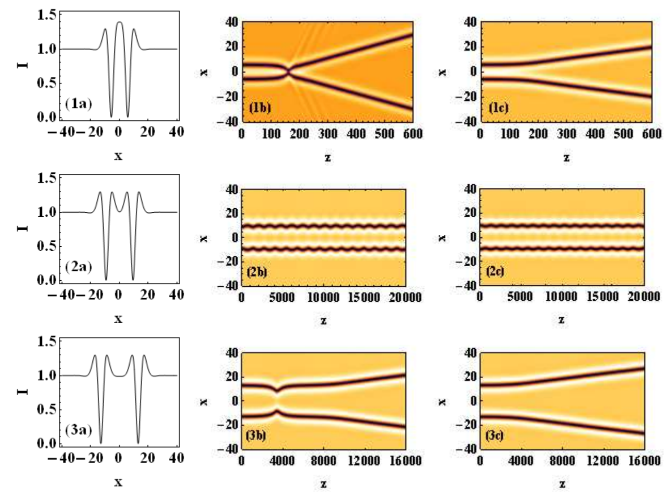

FIG. 6: (1a),(2a),(3a) are intensity profiles of ψ_2, ψ_3, ψ_4; Numerical simulations of initially broadened bound states (1b)$\psi_2(\frac{x}{0.99})$,(1c)$\psi_2(\frac{x}{1.01})$; (2b)$\psi_3(\frac{x}{0.95})$,(2c)$\psi_3(\frac{x}{1.05})$; (3b)$\psi_4(\frac{x}{0.99})$,(3c)$\psi_4(\frac{x}{1.01})$ respectively. Here $w = 5, v = 0$

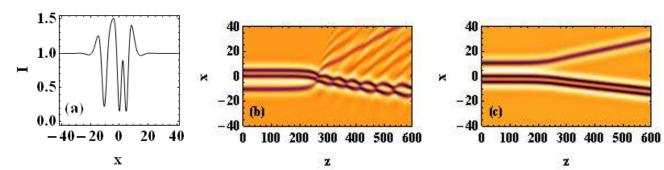

FIG. 7: (a) The intensity profile of $\psi_{2,1}$; Two degenerate bound states (b) $\psi_{2,1}$ and (c) $\psi_{1,2}$ decay in different ways. Here $w = 5, v = 0.25$.

IV. CONCLUSION

In nonlocal media, multiple dark/gray solitons can form bound states in several balance distances. Some of such bound states are unstable and will decay into a group of fundamental solitons, while others may be stable. There exist degenerate bound states with the same velocity, Hamiltonian, particle numbers and momentum but decaying in different ways and having different life-times.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (Grant No. 61008007) and Specialized Research Fund for growing seedlings of the Higher Education in Guangdong (Grant No. LYM10066).

[1] N. I. Nikolov, D. Neshev, W. Krolikowski, O Bang, J. J. Rasmussen, P. L. Christiansen, "Attraction of nonlocal dark optical soliton," Opt. Lett, **29**, 286-288, (2004).

[2] Y. V. Kartashov and L. Torner, "Gray spatial solitons in nonlocal nonlinear media," Opt. Lett, **32**, 946-948, (2007)

[3] S. Ouyang and Q. Guo, "Dark and gray spatial optical solitons in Kerr-type nonlocal media" Optics Express, **17**, 5170, (2009)

[4] I. V. Barashenkov and A. O. Harin, "Topological excitations in a condensate of nonrelativistic bosons coupled to Maxwell and Chern-Simons fields", Phys. Rev D, **52**, 2471, (1995)

[5] I. V. Barashenkov and A. O. Harin, "Nonrelativistic Cherns-Simons theory for the repulsive Bose gas" Phys. Rev. Lett, **72**, 1575, (1993)

[6] Y. S. Kivshar and W. Krolikowski, "Instabilities of dark solitons", Opt. Lett., $bf20$, 1527, (1995)

[7] I. V. Barashenkov, "Stability Criterion for Dark Solitons", Phy. Rev. Lett., **77**, 1193, (1996)

[8] D. E. Pelinovsky, Y. S. Kivshar, V. V. Afanasjev, "Instability-induced dynamics of dark solitons", Phys. Rev E, **54**, 2015, (1996)

[9] X. Gao, L. Zhou, Z. Yang, X. Ma, D. Lu, and W. Hu, "Surface dark solitons in nonlocal nonlinear media", http://arxiv.org/abs/1103.1078v1